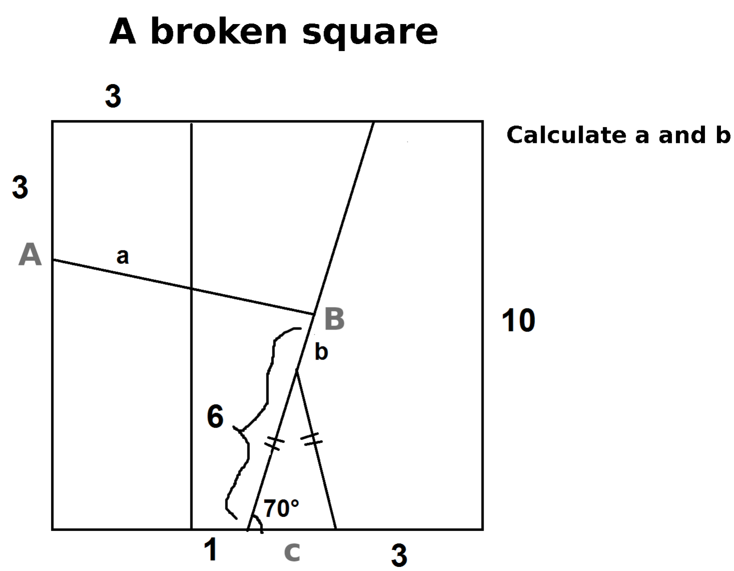
Solutions to geometric problems

Eltjo Maring

September 2020

1 A broken square

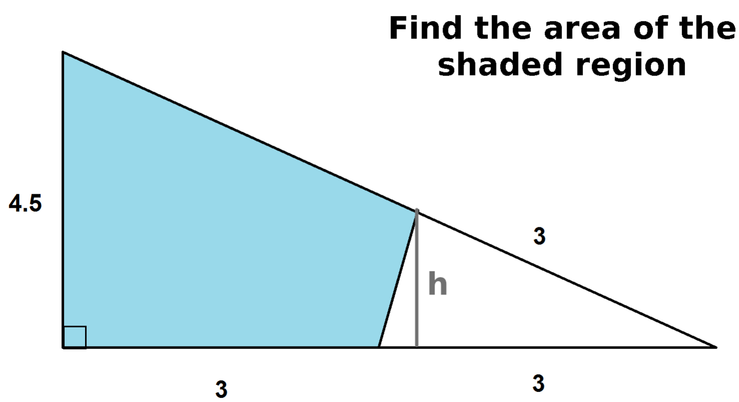


$B = (4 + 6 \cdot \cos(70°), 6 \cdot \sin(70°))$
$A = (0, 10 - 3) = (0, 7)$
Calculating intersection S:
Slope: $s = \frac{B_y - A_y}{B_x - A_x} = \frac{6 \cdot \sin(70°) - 7}{4 + 6 \cdot \cos(70°)}$
$S_y = S_x \cdot s + 7$
$S_x = 3$
$S = (3, \frac{18 \cdot \sin(70°) - 21}{4 + 6 \cdot \cos(70°)} + 7)$
$a = |\overrightarrow{AS}| = \sqrt{(S_x - A_x)^2 + (S_y - A_y)^2} = \sqrt{9 + (3s + 7 - 7)^2} = \sqrt{9 + 9s^2} = 3\sqrt{1 + s^2} = 3\sqrt{1 + (\frac{6 \cdot \sin(70°) - 7}{4 + 6 \cdot \cos(70°)})^2} \approx 3.075$

$\cos(70°) = \frac{\frac{1}{2}c}{6-b}$
$2\cos(70°) = \frac{c}{6-b} = \frac{10-3-3-1}{6-b} = \frac{3}{6-b}$
$6 - b = \frac{3}{2\cos(70°)}$
$b = 6 - \frac{3}{2\cos(70°)} \approx 1.614$

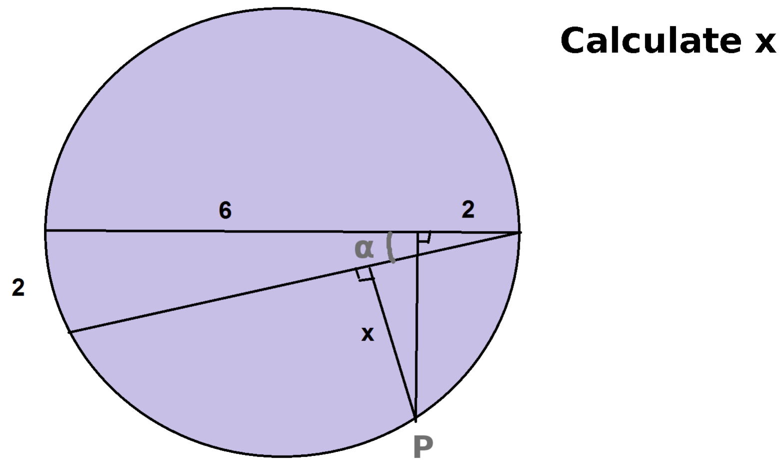
So, in short: $a \approx 3.075$, $b \approx 1.614$

2 Triangle area



Hypotenuse: $s = \sqrt{6^2 + (4\frac{1}{2})^2} = \sqrt{36 + 20\frac{1}{4}} = \sqrt{56\frac{1}{4}} = \sqrt{\frac{225}{4}} = \frac{15}{2} = 7\frac{1}{2}$
$\frac{3}{7\frac{1}{2}} = \frac{h}{4\frac{1}{2}}$ $7\frac{1}{2}h = 3 \cdot 4\frac{1}{2} = 13\frac{1}{2}$
$h = \frac{13\frac{1}{2}}{7\frac{1}{2}} = \frac{27}{15} = \frac{9}{5}$
$A = \textit{Area big triangle} - \textit{Area small triangle} = \frac{1}{2} \cdot 6 \cdot 4\frac{1}{2} - \frac{1}{2} \cdot 3 \cdot \frac{9}{5} = 10\frac{4}{5}$

3 The circle



Assuming the middle line crosses the center point of the circle, we can deduce the necessary information.

$r = \frac{6+2}{2} = 4$

The central angle is twice the inscribed angle α, so $2\alpha = \frac{2}{r} = \frac{2}{4} = \frac{1}{2}$

$\alpha = \frac{1}{4}$

$\vec{v} = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix}$

$\vec{n} = \begin{pmatrix} \sin(\alpha) \\ -\cos(\alpha) \end{pmatrix}$

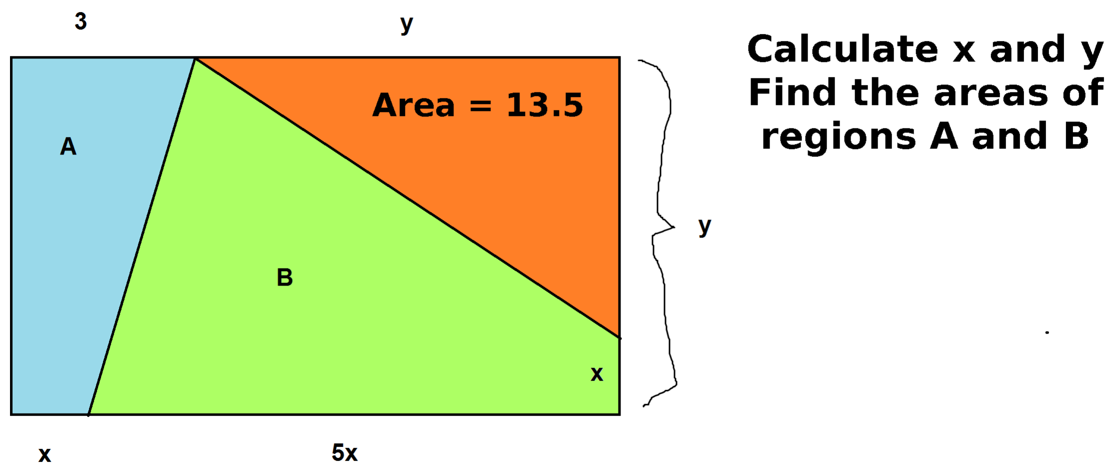
Using the rightmost point as the origin, the coordinates of P can be calculated as follows:

$\vec{P} = \begin{pmatrix} -2 \\ -\sqrt{r^2 - (6-r)^2} \end{pmatrix} = \begin{pmatrix} -2 \\ -\sqrt{16 - (6-4)^2} \end{pmatrix} = \begin{pmatrix} -2 \\ -\sqrt{12} \end{pmatrix} = \begin{pmatrix} -2 \\ -2\sqrt{3} \end{pmatrix}$

$x = \vec{P} \cdot \vec{n} = -2\sin(\alpha) - 2\sqrt{3} \cdot -\cos(\alpha) = -2\sin(\alpha) + 2\cos(\alpha)\sqrt{3}$

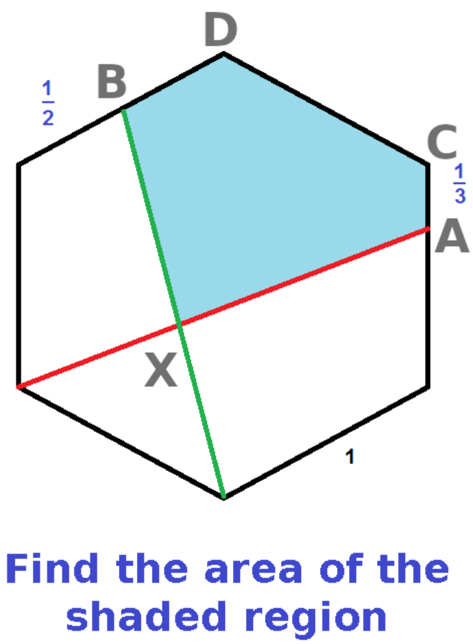
$x = -2\sin(\frac{1}{4}) + 2\cos(\frac{1}{4})\sqrt{3} \approx 2.862$

4 The rectangle



$\frac{1}{2}y(y-x) = 13\frac{1}{2}$
$y(y-x) = 27$
$y + 3 = 6x$
$y = 6x - 3$
$(6x-3)(6x-3-x) = 27$
$(6x-3)(5x-3) = 27$
$30x^2 - 18x - 15x + 9 = 27$
$30x^2 - 33x - 18 = 0$
$10x^2 - 11x - 6 = 0$
$(2x-3)(5x+2) = 0$
$2x = 3 \vee 5x = -2$
$x = 1\frac{1}{2} \vee x = -\frac{2}{5}$
Negative lengths are impossible, so $x = 1\frac{1}{2}$.
$y = 6x - 3 = 6 \cdot (1\frac{1}{2}) - 3 = 9 - 3 = 6$
$A = \frac{x+3}{2} \cdot y = \frac{4\frac{1}{2}}{2} \cdot 6 = 2\frac{1}{4} \cdot 6 = 13\frac{1}{2}$
$B = y(y+3) - A - 13\frac{1}{2} = 6(6+3) - 13\frac{1}{2} - 13\frac{1}{2} = 54 - 27 = 27$

5 Hexagon area



Assuming a perfect hexagon, the coordinates of A,B,C and D can be calculated using the bottom vertex as the origin. Then, the bottom right edge makes an angle of 30° with the x-axis, which means the bottom right vertex has coordinates $(\cos(30°), \sin(30°)) = (\frac{1}{2}\sqrt{3}, \frac{1}{2})$.

$A = (\frac{1}{2}\sqrt{3}, \frac{1}{2} + \frac{2}{3}) = (\frac{1}{2}\sqrt{3}, \frac{7}{6})$

$B = (-\frac{1}{4}\sqrt{3}, \frac{1}{2} + 1 + \frac{1}{4}) = (-\frac{1}{4}\sqrt{3}, \frac{7}{4})$

$C = (\frac{1}{2}\sqrt{3}, \frac{3}{2})$

$D = (0, 2)$

Calculating intersection X:

$y_{green} = \frac{\frac{7}{4}}{-\frac{1}{4}\sqrt{3}}x = -\frac{7}{\sqrt{3}}x = -\frac{7\sqrt{3}}{3}x$

$y_{red} = \frac{\frac{2}{3}}{\sqrt{3}}x + b = \frac{2}{3\sqrt{3}}x + b = \frac{2\sqrt{3}}{9}x + b$

Through $(-\frac{1}{2}\sqrt{3}, \frac{1}{2})$:

$\frac{1}{2} = \frac{2\sqrt{3}}{9} \cdot -\frac{1}{2}\sqrt{3} + b$

$\frac{1}{2} = -\frac{3}{9} + b$

$b = \frac{5}{6}$

$y_{red} = \frac{2\sqrt{3}}{9}x + \frac{5}{6}$

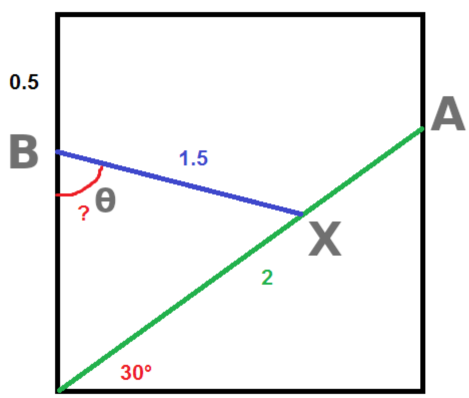
$\frac{2\sqrt{3}}{9}x + \frac{5}{6} = -\frac{7\sqrt{3}}{3}x$

$\frac{23\sqrt{3}}{9}x = -\frac{5}{6}$

$x = -\frac{5}{6} \cdot \frac{9}{23\sqrt{3}} = \frac{-45}{138\sqrt{3}} = -\frac{5\sqrt{3}}{46}$
$y = -\frac{7\sqrt{3}}{3}x = -\frac{7\sqrt{3}}{3} \cdot -\frac{5\sqrt{3}}{46} = \frac{35}{46}$
$X = (-\frac{5\sqrt{3}}{46}, \frac{35}{46})$
Calculating the area:
$\overrightarrow{XA} = \begin{pmatrix} \frac{1}{2}\sqrt{3} + \frac{5\sqrt{3}}{46} \\ \frac{7}{6} - \frac{35}{46} \end{pmatrix} = \begin{pmatrix} \frac{14\sqrt{3}}{23} \\ \frac{28}{69} \end{pmatrix}$
$\overrightarrow{XB} = \begin{pmatrix} -\frac{1}{4}\sqrt{3} + \frac{5\sqrt{3}}{46} \\ \frac{7}{4} - \frac{35}{46} \end{pmatrix} = \begin{pmatrix} -\frac{13\sqrt{3}}{92} \\ \frac{91}{92} \end{pmatrix}$
$\overrightarrow{XC} = \begin{pmatrix} \frac{1}{2}\sqrt{3} + \frac{5\sqrt{3}}{46} \\ \frac{3}{2} - \frac{35}{46} \end{pmatrix} = \begin{pmatrix} \frac{14\sqrt{3}}{23} \\ \frac{17}{23} \end{pmatrix}$
$\overrightarrow{XD} = \begin{pmatrix} 0 + \frac{5\sqrt{3}}{46} \\ 2 - \frac{35}{46} \end{pmatrix} = \begin{pmatrix} \frac{5\sqrt{3}}{46} \\ \frac{57}{46} \end{pmatrix}$
$Area = \frac{1}{2}((\overrightarrow{XA} \wedge \overrightarrow{XC}) + (\overrightarrow{XC} \wedge \overrightarrow{XD}) + (\overrightarrow{XD} \wedge \overrightarrow{XB})) = \frac{1}{2}(\frac{14\sqrt{3}}{23}(\frac{17}{23} - \frac{28}{69}) + \frac{14\sqrt{3}}{23} \cdot \frac{57}{46} - \frac{17}{23} \cdot \frac{5\sqrt{3}}{46} + \frac{5\sqrt{3}}{46} \cdot \frac{91}{92} - \frac{57}{46} \cdot -\frac{13\sqrt{3}}{92}) = \frac{1}{2}(\frac{14\sqrt{3}}{69} + \frac{31\sqrt{3}}{46} + \frac{13\sqrt{3}}{46}) = \frac{1}{2}(\frac{80\sqrt{3}}{69}) = \frac{40}{69}\sqrt{3}$

So, the answer is: $\frac{40}{69}\sqrt{3} \approx 1.004$

6 Calculate the angle



$A = \begin{pmatrix} 2\cos(30°) \\ 2\sin(30°) \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$
$B = \begin{pmatrix} 0 \\ \sqrt{3} - \frac{1}{2} \end{pmatrix}$

$g(t) = At = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} t$
Circle c of radius 1.5, centered at B: $|C - B|^2 = r^2$
$C_x^2 + (C_y - \sqrt{3} + \frac{1}{2})^2 = (\frac{3}{2})^2$
Substituting $C = g(t)$
$(\sqrt{3} \cdot t)^2 + (t - \sqrt{3} + \frac{1}{2})^2 = \frac{9}{4}$
$3t^2 + t^2 - \sqrt{3} \cdot t + \frac{1}{2}t - \sqrt{3} \cdot t + 3 - \frac{1}{2}\sqrt{3} + \frac{1}{2}t - \frac{1}{2}\sqrt{3} + \frac{1}{4} = \frac{9}{4}$
$4t^2 - 2\sqrt{3} \cdot t + t + 3 + \frac{1}{4} - \sqrt{3} = \frac{9}{4}$
$4t^2 + (1 - 2\sqrt{3})t + 1 - \sqrt{3} = 0$
abc-formula, with $a = 4$, $b = 1 - 2\sqrt{3}$, $c = 1 - \sqrt{3}$:
$D = b^2 - 4ac = (1 - 2\sqrt{3})^2 - 4 \cdot 4 \cdot (1 - \sqrt{3}) = 1 - 4\sqrt{3} + 12 - 16 + 16\sqrt{3} = -3 + 12\sqrt{3}$
$t = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 + 2\sqrt{3} \pm \sqrt{-3 + 12\sqrt{3}}}{8} = -\frac{1}{8} + \frac{1}{4}\sqrt{3} \pm \frac{1}{8}\sqrt{-3 + 12\sqrt{3}}$
$-\frac{1}{8} + \frac{1}{4}\sqrt{3} - \frac{1}{8}\sqrt{-3 + 12\sqrt{3}} \approx -0.219 < 0$, so only one positive solution remains:
$t = -\frac{1}{8} + \frac{1}{4}\sqrt{3} + \frac{1}{8}\sqrt{-3 + 12\sqrt{3}} \approx 0.835$
Intersection X:
$X = g(t) = \begin{pmatrix} \sqrt{3} \cdot t \\ t \end{pmatrix} = \begin{pmatrix} \frac{3}{4} - \frac{1}{8}\sqrt{3} + \frac{3}{8}\sqrt{-1 + 4\sqrt{3}} \\ -\frac{1}{8} + \frac{1}{4}\sqrt{3} + \frac{1}{8}\sqrt{-3 + 12\sqrt{3}} \end{pmatrix}$
Now, we just need to find the angle θ between $\overrightarrow{BX}$ and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$:
$|BX| \cos(\theta) = \overrightarrow{BX} \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$\frac{3}{2} \cos(\theta) = (X - B) \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$\cos(\theta) = \frac{2}{3}(B_y - X_y)$
$\theta = \arccos(\frac{2}{3}(\sqrt{3} - \frac{1}{2} + \frac{1}{8} - \frac{1}{4}\sqrt{3} - \frac{1}{8}\sqrt{-3 + 12\sqrt{3}}))$
$\theta = \arccos(-\frac{1}{4} + \frac{1}{2}\sqrt{3} - \frac{1}{12}\sqrt{-3 + 12\sqrt{3}}) \approx 74.66°$